An almost linear stochastic map related to the particle system models of social sciences

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Abstract

We propose a stochastic map model of economic dynamics. In the last decade, an array of observations in economics has been investigated in the econophysics literature, a major example being the universal features of inequality in terms of income and wealth. Another area of inquiry is the formation of opinion in a society. The proposed model attempts to produce positively skewed distributions and the power law distributions as has been observed in the real data of income and wealth. Also, it shows a non-trivial phase transition in the opinion of a society (opinion formation). A number of physical models also generates similar results. In particular, the kinetic exchange models have been especially successful in this regard. Therefore, we compare the results obtained from these two approaches and discuss a number of new features and drawbacks of this model.

1 Introduction

It is known that the distributions of income and wealth possess some robust and stable features which are independent of the economy-specific conditions [1]. But the exact form of the distribution is still debated [2]. It has been a tradition in the economics literature to model the left tail and the mode of the distributions of the incomes with a log-normal [3] distribution and the right tail with a Pareto distribution i.e., a power law [4]. However, a number of recent studies in econophysics literature argue that the left tail and the mode of the distribution is best described by the exponential or gamma distribution and the right tail of the distribution follows a power law (See Ref. [5,6,8]). Also, a significant number of attempts have been devoted to explain the emergence of a consensus in a society and the possibility of a non-trivial phase-transition in the average opinion of the society (See Ref. [25,26]).

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Here, we propose a stochastic map which attempts to produce all of the afore-mentioned features at different limits and in addition, it shows some new features as well. Below, we propose the model and discuss its relative merits and demerits with respect to the kinetic exchange models which also generates similar results. For related literature, see Ref. [1,7,8,15] for theoretical and numerical results on the kinetic exchange models of markets. See Ref. [9,10,11] for detailed analysis of the limit distributions of the solution of random difference equations (of the form $x(t) = a(t)x(t-1) + b(t)$; we will use only a few very particular instances of it). Ref. [12] characterizes gamma distribution which arises from a random difference equation. Ref. [20] was the first paper that suggested that multi-agent interactions in the kinetic wealth exchange models can be simplified and and viewed from from the point of view of a single agent. Ref. [14] was possibly the first paper that connected the stochastic maps and the kinetic exchange models.

2 The map

The Kolkata kinetic wealth exchange models have been successful to generate a realistic description of the income/wealth distributions (See ref. [8]). Many-body interactions are the key ingredients of this class of (kinetic) exchange models. In this paper, we mainly use a single agent framework to discuss the distributional issues. There is no agent-agent interaction in this model. In this regard, our treatment is closer to the *representative* agent paradigm of the modern economics. The specific structure of the market is also not discussed here. We represent the interaction of the agent with the market with a map of the following form,

$$m(t) = min \ \{(\lambda_1 + \epsilon_t\lambda_2)m(t-1) + \xi_t\lambda_3^n, \theta\} \tag{1}$$

where λ_1, λ_2 and λ_3 are linear functions of a single parameter λ with $0 \leq \lambda \leq 1$ such that $0 \leq \lambda_i \leq 1$ for i=1, 2 and 3. We assume that $-\infty \leq n \leq 1$, θ takes value either a positive, finite value (we assume $\theta = 1$, for convinience) or ∞ (or a sufficiently large value) and $\epsilon_t, \xi_t \sim uniform[0, 1]$ and independent, unless specified. In all the simulations, we have assumed $m(0)$ = 1. We denote the time index by subscripts for exogenous random variables (ϵ and ξ) which we shall drop when no confusion arises. While the parameter space is seemingly too large to be considered in details, we shall restrict it considerably by assuming very simple forms of λ_i for all i.

We can interpret it in the following way. An agent has $m(t)$ amount of wealth (or money) at time t of which he saves a random fraction and invests the rest. His return from investment is represented by the additive term $\xi_t\lambda_3^n$. However,

there is an upper limit of $m(t)$ represented by θ. Since we interpret the random multiplicative term as the savings propensity, we assume that $\lambda_1 + \epsilon_t \lambda_2 \leq 1$ for all t i.e., savings propensity is never greater than 1. Our interest lies in finding the pdf (probability density function $P(m)$) in the steady state.

It may be noted that the above map (ignoring θ) has the general form

$$m(t) = a(t)m(t-1) + b(t) \tag{2}$$

which has been studied in great details by Ref. [9,10,11]. In this paper, we have focused on a few particular instances of it for our purpose. Ref. [14] mapped the asset exchange models into 'random iterated function systems'. But it was concerned with the 'yard-sale' model and the 'theft-and-fraud' model (see Ref. [13]) whereas we focus on the CC and CCM models (the models that introduced fixed and distributed savings propensities in the kinetic exchange models; see Ref. [8]) and that is why we have borrowed the basic structure (a fraction of the total wealth is saved and the rest is invested) from the Kolkata wealth exchange models (i.e., the CC and CCM models).

3   Statistical features of the map at some limits

3.1   A positively skewed distribution

We assume $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = 1 - \lambda$, $n = 1$ and $\theta \to \infty$. The equation becomes

$$m(t+1) = \{\lambda + \epsilon(1-\lambda)\}m(t) + \xi(1-\lambda). \tag{3}$$

See Fig. 1 for steady state distributions corresponding to different values of λ. We restrict our attention to the case $n = 1$ as opposed to other positive numbers, for two reasons. One, this choice of n makes the average a constant, independent of n (as the CC and CCM models; see Ref. [8]) and secondly, to simplify the calculations. We will relax this assumption in Section 3.2 where we will assume n is negative and we will see that the exponents of the power law distributions differ for different values of n.

Though we do not know the exact algebraic form of the steady state distributions produced by Eqn. 3, we can find out its moments in order to describe them qualitatively. We can ignore the time index in the steady state. Taking expectations over both sides of Eqn. 3, we get
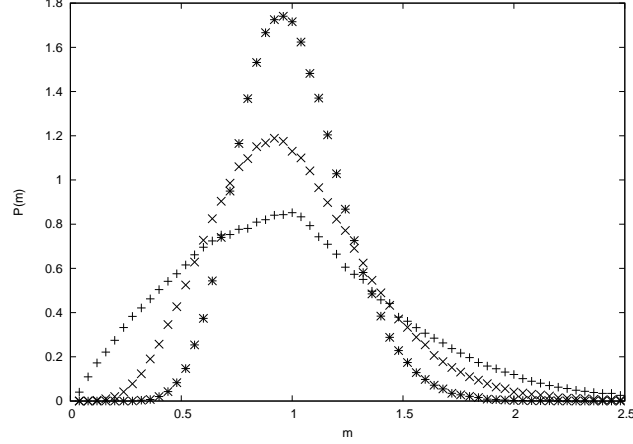
$$\langle m \rangle = 1. \tag{4}$$

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Fig. 1.
Steady state distributions generated by Eqn. 3: Three cases are shown above, viz., $\lambda = 0$ (+), $\lambda = 0.4$ (×), $\lambda = 0.7$ (∗). All simulations are done for $\sim 10^5$ time steps and averaged over for $\sim 10^4$.

Also, we have the variance of m as

$$V(m) = \langle x^2 \rangle - \langle x \rangle^2 \tag{5}$$

where $x = (\lambda + \epsilon(1 - \lambda))\,m + \xi(1 - \lambda)$. Now, we make a few almost trivial observations. Note the fact that $\langle x \rangle = 1$. Hence, $\langle m^2 \rangle$ can be written as $V(m) + \langle m \rangle^2$ (by defn. of $V(m)$) i.e., $V(m) + 1$. Also, ϵ and ξ are uniformly distributed. Therefore, $\langle \epsilon \rangle = 1/2 = \langle \xi \rangle$ and $\langle \epsilon^2 \rangle = 1/3 = \langle \xi^2 \rangle$ (recall that $V(\epsilon) = 1/12 = V(\xi)$). Using all of these and by expanding Eqn. 5 we get

$$V(m) = [\lambda^2 + \lambda(1 - \lambda) + \frac{1}{3}(1 - \lambda)^2]\,(V(m) + 1)$$
$$+ \frac{1}{3}(1 - \lambda)^2 + \frac{1}{2}(1 - \lambda^2) - 1.$$

Simplifying the above expression we get the result for $\lambda \neq 1$,

$$V(m) = \frac{1}{2}\left(\frac{1 - \lambda}{2 + \lambda}\right) \tag{6}$$

which clearly shows that the distribution tends to a delta function as $\lambda \to 1$. With $\lambda = 0$, Eqn. 3 produces a distribution with a sharp peak at m very close to 1. However, for $\lambda > 0.3$ this sharpness goes away.

One can consider an even simpler case with $\xi_t = \epsilon_t$ for all t so that the map

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becomes

$$m(t+1) = \lambda m(t) + \epsilon(1-\lambda)\left(m(t)+1\right) \tag{7}$$

which is almost the same as the usual kinetic exchange model with a constant savings factor i.e., the CC model (See Ref. [27]; See also Ref. [8]),

$$m_i(t+1) = \lambda m_i(t) + \epsilon(1-\lambda)\left(m_i(t)+m_j(t)\right). \tag{8}$$

The variance of the distribution generated by Eqn.7 is given by

$$V(m) = \left(\frac{1-\lambda}{2+\lambda}\right). \tag{9}$$

For the sake of completeness, we mention that the variance of the distribution generated by Eqn. 8 is given by (See Ref. [28])

$$V(m) = \frac{(1-\lambda)}{(1+2\lambda)}. \tag{10}$$

It is noteworthy that for $\lambda = 0$, the distribution generated by Eqn. 7 has a peculiar form which is known as Dickman distiburtion (see Ref. [19]). For $m \leq 1$, $P(m)$ is flat and for $m > 1$, the distribution has a downward slope (see Fig. 2). See Ref. [16], for this type of maps which has been used in number theory, in biology (see Ref. [17]) and in many other areas (see Ref. [18]). See Ref. [19] for simulations on the pdf generated by Eqn. 7 with $\lambda = 0$.

It is to be noted that we cannot generate an exponential distribution from Eqn. 7 since its maximum variance is $1/2$ (for $\lambda = 0$) whereas the variance of the exponential distribution generated by the following Eqn. (See Ref. [8])

$$m_i(t+1) = \epsilon\left(m_i(t)+m_j(t)\right) \tag{11}$$

is unity. Hence this model can not generate purely exponential distribution which is a drawback since recent studies argue that the income/wealth distributions in the real world fits very well with exponential pdfs [1,5,6]. It is trivial to note that in the usual kinetic exchange models, it is the presence of the trading partner's wealth $m_j(t)$ (in the i-th agent's wealth evolution equation; Eqn. 11) that contributes to the higher variance.
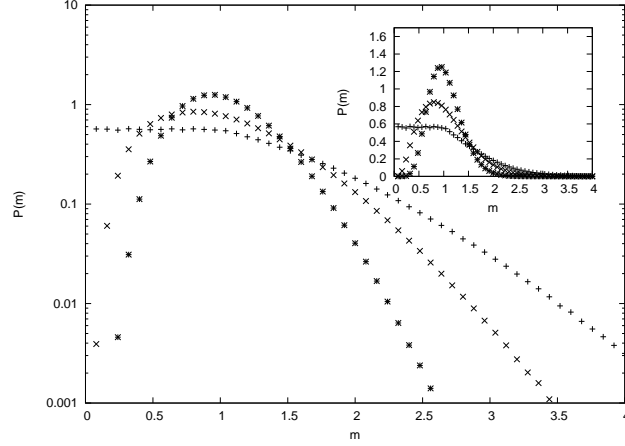
Fig. 2.
Steady state distributions generated by Eqn. 7 are plotted in the semi-log plot : Three cases are shown above, viz., $\lambda = 0$ (+), $\lambda = 0.4$ ($\times$), $\lambda = 0.7$ ($*$). All simulations are done for $\sim 10^5$ time steps and averaged over for $\sim 10^4$. *Inset* : The same data set in the usual plot.

3.2   Power law distribution

We assume $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = 1 - \lambda$, $-\infty \leq n \leq 0$ and $\theta \to \infty$.

The relevant equation is

$$m(t + 1) = \{\lambda + \epsilon(1 - \lambda)\}m(t) + \xi(1 - \lambda)^n. \tag{12}$$

Ref. [22,23] studied this type of discrete stochastic equations (see Eqn. 2) as a generic model for generating power law pdf. But a necessary condition for the mechanism to generate a power law pdf is that the random multiplicative term $a(t)$ ($\lambda + \epsilon(1 - \lambda)$ in Eqn. 12) must be greater than 1 sometimes. In the current context, this condition is not satisfied (according to our interpretation, $(\lambda + \epsilon(1 - \lambda))$ is the savings propensity and hence it is always less than unity). To avoid this problem, we assume a population of agents each of which interacts with the market according to Eqn. 12. Note that the agents do not interact among themselves. In the spirit of the kinetic exchange models, we show that if there is a population of N agents with different λ but each of the income evolution process is modelled by Eqn. 12, then a power law in the income distribution will be observed. However, there is an important difference with the usual kinetic exchange models which are completely conservative. Any trading activity in such markets would be a $zero-sum$ game i.e., if somebody wins then his/her trading partner has to lose. We relax that assumption here. For simulations, we assume that there are 200 agents each endowed with an initial wealth equals to 1. Each of the agent's wealth evolution is governed by
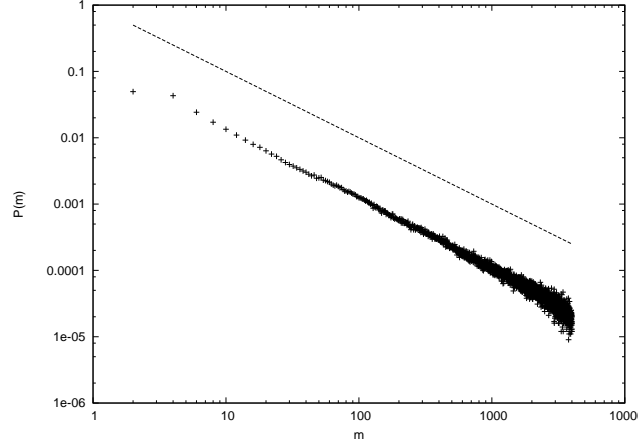
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Fig. 3.
A power law distribution ($n = -20$ in Eqn. 12). The straight line with slope -1 is drawn as a guide. However, here we take $0 \leq \lambda \leq 0.33$ because otherwise the average value of m becomes too large; e.g., Eqn. 13 shows that for $\lambda = 0.5$, $\langle m \rangle = 2^{21}$.
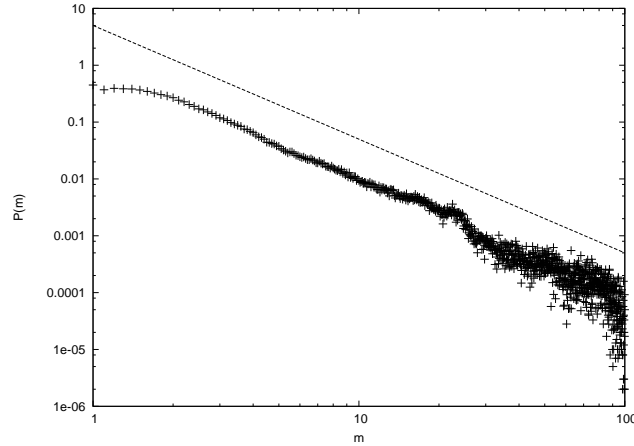

Fig. 4.
A power law distribution ($n = 0$ in Eqn. 12). The straight line with slope -2 is drawn as a guide.

Eqn. 12. Clearly, there is no interactions between the agents. However, the agents are assigned different λs which are fixed over time for any given agent (see Ref. [29]). In particular, we assume that λ is uniformly distributed among the agents. For each agent, simulations are done for $\sim 10^4$ time-steps and the corresponding pdfs are averaged for $\sim 10^3$ time-steps. Resulting distributions are averaged over all agents.

Following Ref. [20], a very simple proof is considered below. Note that (in the steady state) by taking expectations on both sides of Eqn. 12, we can rewrite it as

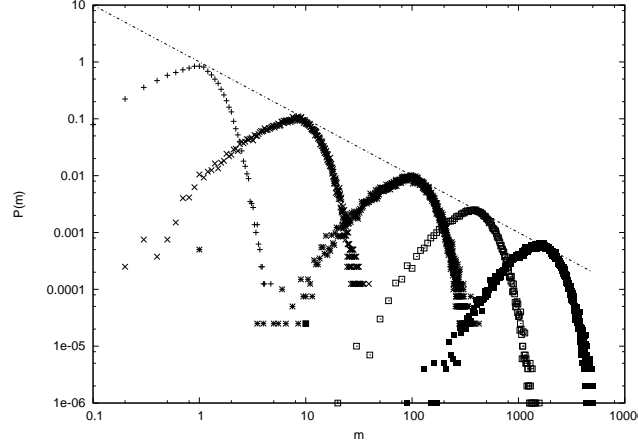$$(1 - \lambda)^{1-n} \langle m \rangle = 1 \qquad (13)$$

Fig. 5.

The pdf of wealth distributions are drawn for the case $n = -20$ with global savings propensities 0, 0.1, 0.2, 0.25, 0.3 (from the left to the right) in the log-log plot. Also, we've drawn m^{-1} in the same diagram (the dotted line).
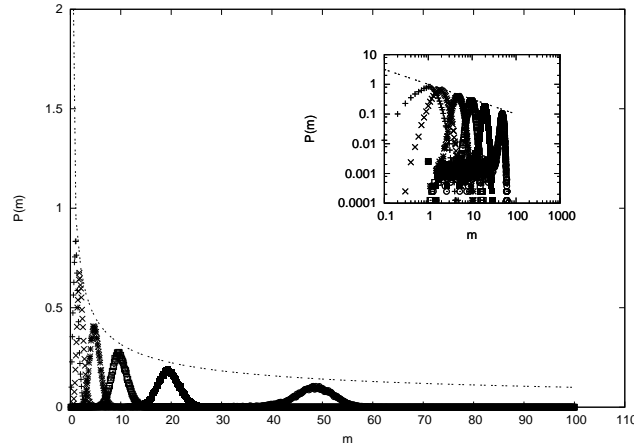

Fig. 6.

The pdf of wealth distributions are drawn for the case $n = 0$ with global savings propensities 0.1, 0.5, 0.8, 0.9, 0.95, 0.98 (from the left to the right). Clearly, the average values of wealth at any given savings propensity are very close to the theoretical values predicted by Eqn. 13. Also, we've drawn $m^{-0.5}$ in the same diagram (the dotted line). *Inset*: The same diagram in log-log plot.

By taking total differentiation and rearrenging terms, we get

$$\frac{d\lambda}{dm} = \frac{1}{1-n} m^{-\frac{n-2}{n-1}} \tag{14}$$

where m represents $\langle m \rangle$. Hence, the average amount of money held by an agent with a particular λ is given by Eqn. 13. Also, the relation between the
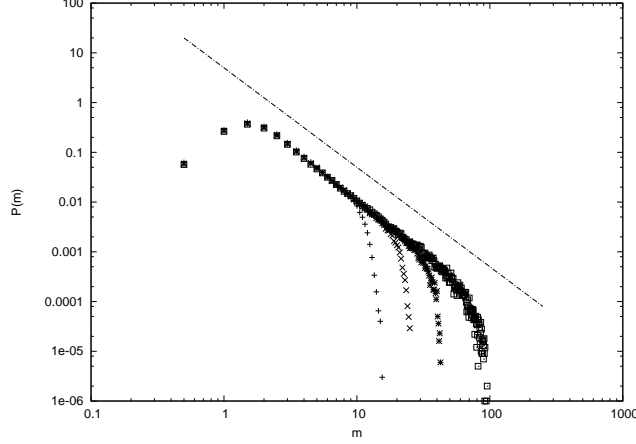
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Fig. 7.
The pdf of wealth distributions are drawn for the case $n = 0$ with maximum savings propensities $\lambda_M = 0.9, 0.95, 0.975, 0.99$ (from the left to the right) in the log-log plot. The savings propensities of the agents are chosen deterministically following the rule that $\lambda_i = (i/N)\lambda_M$. Also, m^{-2} is drawn for reference (the dotted line).

distribution of λ (i.e., $\rho(\lambda)$) with that of m is given by the following Eqn.

$$P(m)dm = \rho(\lambda)d\lambda. \tag{15}$$

Eqn. 14 and 15 shows that in an population of agents with uniformly distributed λ, the distribution of m would be

$$P(m) \sim m^{-\frac{n-2}{n-1}}. \tag{16}$$

It is trivial to note that the CCM model (See Ref. [29]; See also Ref. [8]) derived the result analogous to the case where $n = 0$ (see Fig. 4). Also, if n is large in modulus then the steady state would be power law distribution with power -1 (see Fig. 3) (we take $n = -20$ just for expository purpose).

Following Ref. [8], we can support the above result for the case $n = 0$. Let us rewrite Eqn. 12 with the agent index as the following

$$m_i(t+1) = \{\lambda_i + \epsilon_i(1-\lambda_i)\}m_i(t) + \xi_i(1-\lambda_i)^n. \tag{17}$$

which can be rewritten (using Eqn. 13) as

$$m_i(t+1) = \{\lambda_i + \epsilon_i(1-\lambda_i)\}m_i(t) + \xi_i(1-\lambda_i)\langle m_i \rangle. \tag{18}$$

The above equation holds true for all agents i. Since we focus on average money holding at different savings propensities, we randomly pick any two

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agents i and j (with savings propensities λ_i and λ_j respectively) and multiply their corresponding wealth evolution equations (18), to get

$$
\begin{aligned}
m_i(t+1)m_j(t+1) = {} & f_{ij}^1 m_i(t)m_j(t) + f_{ij}^2 \langle m_i(t) \rangle m_j(t) \\
& + f_{ij}^3 m_i(t) \langle m_j(t) \rangle + f_{ij}^4 \langle m_i(t) \rangle \langle m_j(t) \rangle
\end{aligned}
\tag{19}
$$

where f^ks are functions of ϵ, ξ, λ_i and λ_j for all k. We approximate each of the quadratic quantities by a mean quantity m^2. Note, that here we assumed that m_i can be replaced by its average value $\langle m_i \rangle$, which holds true only if its variance is small and that requires n to be as small in modulus as possible in Eqn. 17. We consider $n = 0$. Then, we have the following equation

$$
m^2(t+1) = \alpha(t+1)m^2(t) \qquad \text{where } \alpha \text{ is a function of } f^k\text{s. .}
\tag{20}
$$

Ref. [21,8] studied this map and showed that as $t \to \infty$,

$$
P(m) \sim m^{-2}.
\tag{21}
$$

Note that this argument holds only if $n = 0$.

We consider the cases where all agents have the same savings propensities which we call the global savings propensities (this case is almost identical to the one described in Section 3.1 in that we consider agents who have the same λ; the difference is that there we considered only one agent whereas here we consider N identical agents. Also, another obvious difference is that here $n \leq 0$). The pdfs are drawn for individual savings propensities in Fig. 5 and 6. Let m have a distribution $f(m)$ in the steady state (given λ). With a slight abuse of notation, we write the value of $f(.)$ evaluated at $m = \bar{m}$ as $f(\bar{m})$. Then Fig. 5 shows that (for $n = -20$) given λ at $m = \langle m \rangle$, $f_\lambda(\langle m \rangle) \approx \langle m \rangle^{-1}$. Similarly, for $n = 0$ given λ at $m = \langle m \rangle$, $f_\lambda(\langle m \rangle) \approx \langle m \rangle^{-0.5}$. This shows that in the case where n is large in modulus, the $f_\lambda(m)$ curve is a good approximation of the average of all the individual wealth distributions which is not true in the other case where $n = 0$. The reason is that the probability of finding an agent within a specified interval of wealth decreases very rapidly with increase in wealth in the case where $n = 0$ compared to the other case (this can be checked very easily from Eqn. 13). Hence, though the uppermost points form a locus of $f_\lambda(\langle m \rangle) \approx \langle m \rangle^{-0.5}$ for $n = 0$, the average distribution falls much more rapidly following $P(m) \sim m^{-2}$.

We have also studied the effects of the maximum λ present in the population on the steady state distribution (for the case $n = 0$). Following Ref. [24], we the assign the savings propensities to the agents deterministically, according to the rule that the i-th agent's savings propensity $\lambda_i = (i/N)\lambda_M$ where N

is the number of agents in the population, $\lambda_M < 1$ is the maximum savings propensity in the population. See Fig. 7 for the results. Curves from left to right represent the distributions for increasing λ_M. Clearly, as λ_M decreases, the power law interval shrinks. Therefore, higher savings propensities contribute to the formation of the power law interval. This conclusion is consistent with the findings in Ref. [24]. We should also mention that according to Eqn. 13, higher λ implies higher average wealth at that particular λ. This fact also supports the findings in Fig. 7.

3.3   A non-trivial phase transition

Here, we briefly mention that a special case of the map shows a non-trivial phase transition. If we assume $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = 0$, $n = 1$ and $\theta = 1$. The equation becomes

$$m(t+1) = min\{\lambda(1+\epsilon)m(t), 1\} \tag{22}$$

This map has been used to model opinion formation. Assume that an agent can have an opinion $m(t)$ within 0 and 1 at any time t. λ is the agent's conviction parameter. After each interaction with the society, his opinion changes by a random fraction. However, the maximum value of opinion that he can have, is 1. Ref. [26] considered this map and showed that this map shows a phase transition (with respect to λ) in the average value of m. The critical value of λ found from simulations, is $\lambda_c \approx 0.68$ and it only mentions an analytical result that $\lambda_c \approx 0.6796$. For numerical details on this particular map, see Ref. [26]. Interestingly, this map has a parallel in the kinetic exchange models, also discusses in the same reference (which has a critical point at $\lambda_c \approx 0.6667$). For detailed analysis on the nature of such transitions see Ref. [30,31].

Let us now focus on the cases with $\lambda < 1$. Consider the map

$$m(t+1) = \lambda(1+\epsilon)m(t). \tag{23}$$

Let us ignore $m(t)$ for all $t \leq \tau$ where τ is sufficiently large. For any time step $T \geq \tau$,

$$m(T) = \lambda^{T-\tau}(1+\epsilon_T)(1+\epsilon_{T-1})...(1+\epsilon_{T-\tau+1})m(\tau). \tag{24}$$

Following Ref. [26], we can argue that for $\lambda < \lambda_c$, the product of all the terms multiplied to $m(\tau)$ in the above equation is less than unity on average. But this term tends to 1 as $\lambda \to \lambda_c$. Therefore at the critical point ($\lambda = \lambda_c$), by

taking logs on both sides we can write

$$- \log \ \lambda_c = \frac{1}{T - \tau} \sum_k^T \log \ (1 + \epsilon_k). \tag{25}$$

As $T \to \infty$, we apply the LLN and the r.h.s. of the above Eqn. converges to $\langle \log \ (1 + \epsilon) \rangle$. By applying Jensen's inequality, we see that

$$- \log \ \lambda_c < \log(\langle 1 + \epsilon \rangle) \tag{26}$$

i.e., $\lambda_c > 2/3$. Therefore the critical point of the map model is greater than that of its kinetic exchange version. By numerical calculations from Eqn. 25, we see that $\lambda_c \approx 0.67954$ which is very close to the value reported in Ref. [26]. One could also find that $\langle \log \ (1 + \epsilon) \rangle = 2 \log 2 - 1$ and therefrom find λ_c using Eqn. 25 (see [26]).

It is trivial to note that for $\lambda < \lambda_c$, $m(t)$ tends to 0. Also at $\lambda = \lambda_c$, the distribution has an abrupt change in variance as has been shown numerically in Ref. [26]. Here, we present a very short proof. By Eqn. 24, $\langle m(T) \rangle = \langle m(\tau) \rangle$ at $\lambda = \lambda_c$ and hence for $T - \tau$ sufficiently large,

$$m(T) - \langle m(T) \rangle \approx \lambda_c^{T-\tau}(1 + \epsilon_T)(1 + \epsilon_{T-1})...(1 + \epsilon_{T-\tau+1})(m(\tau) - \langle m(\tau) \rangle). \tag{27}$$

By squaring and taking expectations on both sides, we get

$$\begin{aligned} V(m(T)) &\approx \langle \lambda_c^{2(T-\tau)}(1 + \epsilon_T)^2(1 + \epsilon_{T-1})^2...(1 + \epsilon_{T-\tau+1})^2 \rangle V(m(\tau)) \\ &> \langle \lambda_c^{T-\tau}(1 + \epsilon_T)(1 + \epsilon_{T-1})...(1 + \epsilon_{T-\tau+1}) \rangle^2 V(m(\tau)) \\ &= V(m(\tau)). \end{aligned} \tag{28}$$

To get the last inequality, note that $V(x) = \langle x^2 \rangle - \langle x \rangle^2 > 0$ for all x (unless it is the trivial case that $x = \langle x \rangle$). Therefore, $V(m(T)) > z_{(T-\tau)} V(m(\tau))$ with $z_k > 1$ for all k and for all τ. Hence, as $t \to \infty$, $V(m(t)) \to \infty$ at $\lambda = \lambda_c$.

To illustrate the problem of non-stationarity, we focus on another particular case where $\lambda \to 1$, $\epsilon \sim uniform[0, \bar{\epsilon}]$ with $\bar{\epsilon} \to 0$ such that $\langle \epsilon \rangle = - \log \ \lambda$ i.e., $\langle \epsilon \rangle = - \log \ (1 - (1 - \lambda)) \approx 1 - \lambda$. From Eqn. 24, by taking logs we get

$$\begin{aligned} \log \ m(T) &= (T - \tau) \log \ \lambda + \sum_k^T \log \ (1 + \epsilon_k) + log \ m(\tau) \\ &\approx \sum_k^T (\epsilon_k - \langle \epsilon_k \rangle) + \log \ m(\tau). \end{aligned} \tag{29}$$

Multiplying both sides by $1/\sqrt{T}$ and by applying CLT (assuming $T \to \infty$), we get

$$\frac{1}{\sqrt{T}} \log \; m(T) \; \to \; N(0, \sigma^2) \qquad \text{in distribution,} \tag{30}$$

where σ^2 is the variance of ϵ. Hence, $m(T)$ is distributed log-normally. But evidently, the distribution is not stationary.

3.4   The law of proportionate effect

We assume $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$ and $\theta \to \infty$. Note that this is the limit of the case discussed above with $\lambda = 1$ and θ sufficiently large. In this case, the equation becomes

$$m(t+1) = (1 + \epsilon_{t+1})m(t) \tag{31}$$

which is very wellknown as a generator of log-normal distribution (See Ref. [3]). To find the distribution of $m(t)$, note that the above equation can be rewritten (by iteration) as

$$m(t+1) = \prod_{k}^{t}(1 + \epsilon_k)m(1) \tag{32}$$

and by taking log on both sides, it can be written as

$$\begin{aligned}
\log \; m(t+1) &= \sum_{k}^{t} \log \; (1 + \epsilon_k) + log \; m(1) \\
&\approx \sum_{k}^{t} \epsilon_k + \log \; m(1) \qquad \text{(for ϵ very small).}
\end{aligned} \tag{33}$$

By applying the Central Limit Theorem, we see that $\log \; m(t)$ is distributed normally, hence $m(t)$ is distributed log-normally. However, evidently the distribution generated this way is not stationary either.

4 Summary

In recent years, a number of economic regularities have been investigated in the econophysics literature, one of the most prominent themes being the distributions of income/wealth. The two candidate distributions for explaining the left tail and the mode of the distributions of income/wealth are Gamma and Log-normal. There is a consensus that the heavy right tail is best described by a power law (See Ref. [1,8]). Also, the process of opinion formation due to interactions among numerous agents have been studied in details. The phenomena of emerging consensus and a phase transition in the opinion formation have been tried to be modelled in many ways (See Ref. [25,26]).

In this paper, we examine a stochastic map to model the economic process of income/wealth distribution and the sociological process of opinion formation. We showed that this particular map can generate a positively skewed pdf and a power law pdf. It shows a non-trivial phase-transition and in another limit, it coincides with a very well known generator of log-normal pdf. However, there are several drawbacks of this approach. One is that the algebraic form of the pdf generated by the Eqn. 3 is not known and it is not a Γ pdf (at one limit it has a sharp peak which eventually goes away). Second is the well known problem with Eqn. 31 that the distribution generated by this equation is not stationary. Third, purely exponential distribution cannot be generated by this model. But there are a number of recent studies concluding that the income/wealth distribution can be modelled by exponential pdf [1,5,6]. Previously, the kinetic exchange models have been successful to model many of these phenomena [8,26]. We show that there are some new results produced by this map model whereas other results mostly conform with those derived in the kinetic exchange models.

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